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On the Stability of Integral Manifolds of Functional Differential Equations*

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INTRODUCTION

In a previous paper [6], the author established (under appropriate assumptions) the asymptotic orbital stability of an isolated periodic solution (limit cycle) of an autonomous functional differential equation (FDE, henceforth).

In the present work, this result (stated in Section 2) is extended to a $(k+1)$-parameter family (integral manifold) of periodic solutions of an autonomous FDE. The method of approach is similar to that used previously: First, an asymptotically stable cross-section of the flow in the neighborhood of the manifold is obtained by solving a nonlinear integral equation (Section 3); second, this cross-section is shown to be transverse to the flow in some neighborhood of the manifold (Section 4). This last proof uses the standard implicit function theorem in Banach spaces. To apply such a theorem requires a certain amount of information regarding the differentiability of solutions of an FDE with respect to initial conditions and parameters. These results are presented in Section 1, and are the expected extensions of the corresponding theorems in ordinary differential equations.

In a subsequent paper Ref. [8], the author has investigated the stability of such an integral manifold under various classes of perturbations. This completes the extension to functional differential equations of the results obtained in Ref. [3] for ordinary differential equations.

1. DEFINITION 1.1. Let C denote the space of continuous functions from $[-h, 0] \to R^n$, $h > 0$, with the norm defined by $\|\phi\| = \sup\|\phi(\theta)\|$, $-h \leqslant \theta \leqslant 0$, $\|\phi(\theta)\|$ any norm in R^n. Note that this same symbol for a norm, namely $\|\ \|$, will be used in all spaces, as the context will make clear which norm is meant. It is well-known that C in the given norm is a Banach space.

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DEFINITION 1.2. If $w : [-h, A) \to R^n$ is a continuous function, then w determines a map $w^* : [0, A) \to C$ in the following fashion: Define w^* by $w_t^*(\theta) = w(t + \theta)$, $-h \leqslant \theta \leqslant 0$, $0 \leqslant t < A$. Here w_t^* denotes the element of C given by the map w^* at t. Then $w_t^*(\theta)$ denotes the value in R^n of w_t^*, for $\theta \in [-h, 0]$.

In the sequel, the distinction between the function $w : [-h, A) \to R^n$, and the function $w^* : [0, A) \to C$, will be clear from the context, so both functions will be denoted by w. w_t will then be used to denote an element of C, while $w(t)$ denotes a vector in R^n.

In the following, let E and F denote two Banach spaces.

DEFINITION 1.3. Given a function $f : E \to F$, we will say that f is Fréchet differentiable at $\phi \in E$ (see Vainberg [7], Chap. I) if there exists a bounded linear map $L : E \to F$ such that $\| f(\phi + \Psi) - f(\phi) - L(\Psi)\| = o(\| \Psi \|)$, as $\| \Psi \| \to 0$, for all $\Psi \in E$.

We will call L the Fréchet derivative of f at ϕ. If f has a Fréchet derivative for every ϕ in some ball M in E, we will write $df(\phi; \Psi)$ to denote the value of the Fréchet derivative of f at ϕ, evaluated at Ψ.

Thus, we have a mapping $d : M \to L(E, F)$, where $L(E, F)$ is the space of bounded linear operators from $E \to F$, taking $\phi \to df(\phi; \cdot)$. $L(E, F)$ is a normed space, in the usual operator norm. The statement that $df(\phi; \Psi)$ is continuous in ϕ for $\phi \in M$ will then mean that the mapping d above is continuous with respect to the usual norm on E, and the operator norm on $L(E, F)$.

It follows from the definition of the Fréchet derivative that if f has a Fréchet derivative at ϕ, then for $\Psi \in E$,

$$f(\Psi) = f(\phi) + df(\phi; \Psi - \phi) + g(\phi, \Psi - \phi), \tag{1.1}$$

where $\| g(\phi, \Psi - \phi)\| = o(\| \Psi - \phi \|)$ as $\| \Psi - \phi \| \to 0$.

Actually, when f has a continuous Fréchet derivative, an even more precise relationship exists between $f(\phi)$ and $f(\Psi)$.

LEMMA 1.4. *Let E, F be two Banach space. Let M be an open convex subset of E, and let $f : M \to F$ be a mapping with a Fréchet derivative defined and continuous on M. Let $df(\phi; \cdot)$ denote the value of this derivative at ϕ. Then, for any $\phi, \Psi \in M$, we have*

$$f(\Psi) - f(\phi) = \int_0^1 df(s\Psi + (1 - s)\phi; \Psi - \phi)\, ds, \tag{1.2}$$

(*where the integral is to be taken in the usual sense as the limit in the F-norm of Riemann sums formed from the integrand, see Dieudonné* [2].

Proof. Lemma 1.4 is Taylor's formula, in the case $p = 1$, given in Theorem 8.14.3 of Dieudonné [2].

DEFINITION 1.5. Given a function $f : E \to F$, if for every $\alpha \in E$,

$$\lim_{r \to 0} \frac{f(\phi + r\alpha) - f(\phi)}{r} = Vf(\phi; \alpha)$$

exists, then the operator $Vf(\phi; \alpha)$ is called the Gateaux differential of f at ϕ in the direction α.

In general, $Vf(\phi; \alpha)$ need not be linear in α. If $Vf(\phi; \alpha)$ is a continuous linear operator in α, we will call it the Gateaux derivative of f at ϕ, and write $Df(\phi; \alpha)$. Clearly, the existence of the Fréchet derivative at ϕ implies the existence of the Gateaux derivative at ϕ. The following lemma gives sufficient conditions for the converse.

LEMMA 1.6. *Given a function $f : E \to F$, if the Gateaux derivative $Df(\phi; \cdot)$ exists in some neighborhood U of ϕ, and if $Df(\phi; \cdot)$ is continuous at ϕ (as a mapping from $U \to L(E, F)$), then the Fréchet derivative of f at ϕ exists, and $df(\phi; \cdot) = Df(\phi; \cdot)$.*

Proof. See Vainberg [7, p. 41].

DEFINITION 1.7. A FDE is an equation of the form

$$\dot{x}(t) = f(t, x_t), \tag{1.3}$$

where $f : [t_0, t_1) \times C \to R^n$ is continuous in both variables, and $\dot{x}(t)$ denotes the right-hand derivative of x at t, for $t_0 \leqslant t < t_1$.

DEFINITION 1.8. Given $s \in [t_0, t_1)$; by a solution of (1.3) is meant a continuous map $x : [s - h, s + A) \to R^n$, such that the right-hand derivative of $x(t)$ exists for $s \leqslant t < s + A$ and (1.3) is satisfied identically on $[s, s + A)$. Here $0 < A \leqslant t_1 - s$.

The following theorem is well-known (see Krasovskii [5]).

LEMMA 1.9. *Assume $f(t, \phi)$ satisfies a Lipschitz condition in ϕ, with Lipschitz constant L, for $\| \phi \| \leqslant H$, and $t_0 \leqslant t < t_1$. Then for any $s \in [t_0, t_1)$ and any $\phi \in C$, $\| \phi \| < H$, there exists a unique solution of (1.3), denoted by $x_t(\phi, s)$; defined on some interval $[s, s + A)$, with $x_s(\phi, s) = \phi$. In addition, given $A_1 < A$, there exists a $\gamma > 0$ such that if $\| \phi_1 - \phi \| < \gamma$, $\phi_1 \in C$, then $x_t(\phi_1, s)$ is defined on $[s, s + A_1]$, and on that interval $\| x_t(\phi, x) - x_t(\phi_1, s)\| \leqslant e^{L(t-s)} \| \phi - \phi_1 \|$.*

Lemma 1.9 asserts that the solutions of (1.3) depend continuously on their initial conditions. The following lemma is a similar statement concerning the dependence of solutions on parameters.

LEMMA 1.10. *Consider the equation*

$$\dot{x}(t) = f(t, x_t, \mu), \tag{1.4}$$

where $f : [t_0, t_1) \times C \times U \to R^n$ *is continuous is the arguments* (t, ϕ, μ), *where* $\mu \in U$, U *an open subset of some metric space* S *with metric* ρ. *Assume* f *satisfies a Lipschitz condition in* ϕ *for* $\|\phi\| \leqslant H$, *uniformly in* $t \in [t_0, t_1)$, $\mu \in U$. *For* $s \in [t_0, t_1)$, $\phi \in C$, $\|\phi\| < H$, $\mu_0 \in U$, *let* $x_t(s, \phi, \mu_0)$ *be defined on some interval* $[s, s + A)$. *Then, given* $\beta > 0$, *and* $A_1 < A$, *there exists a* $\gamma > 0$ *such that if* $\phi_1 \in C$, $\mu \in U$, $\|\phi_1 - \phi\| < \gamma$, $\rho(\mu, \mu_0) < \gamma$, *then* $x_t(s, \phi_1, \mu)$ *is defined on* $[s, s + A_1]$ *and* $\| x_t(s, \phi_1, \mu) - x_t(s, \phi, \mu_0)\| < \beta$ *for* $t \in [s, s + A_1]$.

Proof. The proof can be obtained as a straightforward modification of the proof of Theorem 7.4, p. 29 in Coddington and Levinson [1]. The only change needed is to replace the requirement that f be uniformly continuous on some neighborhood of the set $(t, x_t(s, \phi, \mu_0), \mu_0)$ in $[s, s + A_1] \times C \times U$ by use of the assumption that f satisfies a Lipschitz condition in ϕ uniformly on $[s, s + A_1] \times U$, and the fact that $f(t, x_t(s, \phi, \mu_0), \mu) \to f(t, x_t(s, \phi, \mu_0), \mu_0)$ as $\mu \to \mu_0$ uniformly in t on $[s, s + A_1]$.

Remark. Lemma 1.10 implies that a solution $x_t(s, \phi)$ of (1.3) is continuous in s, for $t_0 < s < t_1$. To see this, introduce a change of variable $t = s + \tau$, which changes (1.3) to

$$\dot{x}(\tau) = f(s + \tau, x_\tau), \qquad \text{with} \quad x_0(0, \phi) = \phi. \tag{1.5}$$

In (1.5), s appears as a parameter, and so Lemma 1.10 implies that solutions of (1.5), and corresponding of (1.3), are continuous in s. Of course, as solutions of (1.3) cannot be continued to the left, two solutions can only be compared on their common domain of definition.

The following theorem gives an additional result.

THEOREM 1.11. *Let* $f(t, \phi)$ *be continuous on the set*

$$N = [t_0, t_1) \times \{\phi \in C : \|\phi\| < H\}$$

and let f *possess a Fréchet derivative* $df(t, \phi; \cdot)$ *with respect* ϕ *continuous in* (t, ϕ) *on* N. *Take* $(s, \phi_0) \in N$, *and let* $x_t(s, \phi_0)$ *be the solution of* (1.3) *defined on*

$[s, s + A]$, *with* $x_s(s, \phi_0) = \phi_0$: (i) *Then* $x_t(s, \phi_0)$ *has a Fréchet derivative* $dx(t, s; \phi_0; \cdot)$ *with respect to* ϕ_0, *which is continuous on its domain of definition* $s < t < s + A$, $(s, \phi_0) \in N$. (ii) *Furthermore, for* $\alpha \in C$, $dx(t, s, \phi_0; \alpha) = z_t(s, \alpha)$, *where* $z_t(s, \alpha)$ *is the solution of the linear* FDE:

$$\dot{z}(t) = df(t, x_t(s, \phi_0); z_t), \qquad \textit{with} \quad z_s(s, \alpha) = \alpha. \tag{1.6}$$

(*The following proof is adapted from that appearing in Hartman* [4, p. 95 ff].

Proof. By Lemma 1.6, it suffices to show that: (a) Given $\alpha \in C$ the

$$\lim_{r \to 0} \frac{1}{r} [x_t(s, \phi_0 + r\alpha) \quad x_t(s, \phi_0)] \text{ exists, for each } t \in [s, s + A). \tag{1.7}$$

(b) The limit in (1.7) is a continuous linear function of α, and thus it is the Gateaux derivative $Dx(t, s, \phi_0; \cdot)$. (c) $Dx(t, s, \phi_0; \cdot)$ is continuous in (t, s, ϕ_0), for $s \leqslant t < s + A$, $(s, \phi_0) \in N$.

This will complete the proof of (i). The proof of (ii) will appear in the course of proving (a), (b), and (c) above.

Let $0 < A_1 < A$ be arbitrary. Clearly it will suffice to establish (a), (b), and (c) on the interval $[s, s + A_1]$.

Take $\alpha \in C$. Let $x_t^r = x_t(s, \phi_0 + r\alpha)$, let $x_t^0 = x_t(s, \phi_0)$. For $| r |$ sufficiently small, Lemma 1.9 asserts that x_t^r is defined on $[s, s + A_1]$, and

$$x_t^r \to x_t^0 \qquad \text{as} \quad r \to 0 \tag{1.8}$$

uniformly on $[s, s + A_1]$. Now, applying Lemma 1.4, from (1.3) it follows that

$$\begin{aligned} \dot{x}^r(t) - \dot{x}^0(t) &= f(t, x_t^r) - f(t, x_t^0) \\ &= \int_0^1 df(t, ux_t^r + (1 - u)\, x_t^0; x_t^r - x_t^0)\, du, \\ &\equiv J(t, s, \phi_0, r; x_t^r - x_t^0), \end{aligned} \tag{1.9}$$

where the last line is a definition of the operator $J(t, x, \phi_0, r; \beta)$, which by the assumptions on $df(t, \phi; \cdot)$ is continuous in all its arguments, and linear in β, for $\beta \in C$. Further, note that from (1.8) and (1.9), it follows that $J(t, s, \phi_0, r; \cdot) \to df(t, x_t(s, \phi_0); \cdot)$ as $r \to 0$, uniformly on $[s, s + A_1]$.

Let

$$z_t^r = \frac{1}{r}(x_t^r - x_t^0). \tag{1.10}$$

Note that (a) is equivalent to the statement $\lim_{r \to 0} z_t^r$ exists.

From (1.9) and (1.10), it follows that x_t^r is the solution of the linear FDE

$$\dot{z}(t) = J(t, s, \phi_0, r; x_t), \qquad \text{with} \quad z_s = \alpha. \tag{1.11}$$

If (1.11) is regarded as a linear FDE with a scalar parameter r, Lemma 1.10 asserts that for each sufficiently small r, the solution z_t^r exists and is unique over the interval $[s, s + A_1]$. Further, Lemma 1.10 asserts that z_t^0 exists and is the solution of (1.6) satisfying $z = \alpha$. Thus, part (a) has been proven.

But as (1.6) is a linear FDE, evidently $z_t(s, \alpha)$ is linear in α, for each t, and from Lemma 1.9, a continuous function of α. This proves (b).

But if (s, ϕ_0) are regarded as parameters in (1.6), then continuity assumption on $df(t, \phi; \cdot)$ together, with Lemma 1.10 imply that $z_t(s, \alpha)$ depends continuously on s, ϕ_0 (and, of course, on t) uniformly in α, $\| \alpha \| = 1$. This establishes (c) and from Lemma 1.6, we conclude that the Fréchet derivative $dx(t, s, \phi_0; \cdot)$ exists with the properties given in (i). The assertions in (ii) are already clear from the above proof.

COROLLARY 1.12. *Consider the equation*

$$\dot{x}(t) = f(t, x_t, \Psi), \tag{1.12}$$

where $f(t, \phi, \Psi)$ is continuous on the set $N_1 = [t_0, t_1) \times \{\phi \in C : \| \phi \| < H\} \times V$, where V is an open convex subset of a Banach space E. Further, assume f possesses Fréchet derivatives $d_1 f(t, \phi, \Psi; \cdot) \in L(C, R^n)$, with respect to ϕ, and $d_2 f(t, \phi, \Psi; \cdot) \in L(E, R^n)$, with respect to Ψ, which are continuous on N_1. Take $(s, \phi_0, \Psi_0) \in N_1$, and let $x_t(s, \phi_0, \Psi_0)$ be the solution of (1.12) *with $\Psi = \Psi_0$, defined on $[s, s + A)$, with $x_s(s, \phi_0, \Psi_0) = \Psi_0$. Then $x_t(s, \phi_0, \Psi_0; \cdot)$ has a Fréchet derivative $d_2 x(t, s, \phi_0, \Psi_0; \cdot) \in L(E, C)$ with respect to Ψ_0 which is continuous on the domain $s \leqslant t < s + A$, $(s, \phi_0, \Psi_0) \in N_1$. Furthermore, for $\beta \in E$, $d_2 x(t, s, \phi_0, \Psi_0; \cdot) = w_t(s, \beta)$, where $w_t(s, \beta)$ is the solution of the non-homogeneous linear* FDE

$$\dot{w}(t) = d_1 f(t, x_t(s, \phi_0, \Psi_0), \Psi_0; w_t) + d_2 f(t, x_t(s, \phi_0, \Psi_0), \Psi_0; \beta),$$
$$\textit{with} \quad w_s(s, \beta) = 0. \tag{1.13}$$

Proof. The proof parallels the proof of Theorem 1.11, with the obvious modifications.

Remark. If Corollary 1.12 is applied to the special case in which the parameter Ψ belongs to a finite-dimensional Banach space E, a different form may be given to the conclusion of Corollary 1.12, which will clarify the connection between Theorem 1.11, Corollary 1.12, and the corresponding standard results in ordinary differential equation (e.g., [4, Theorem 3.1, p. 95]. The change in the conclusion when E is finite dimensional is due to the fact that C is isometric to the space $L(R, C)$. Accordingly, if we let $E = R^k$, and write $b = (b^1 \cdots b^k) \in E$ instead of Ψ, and introduce the notation

$$\frac{\partial x(t, s, \phi_0 ; b_0)}{\partial b^i} = d_2 x(t, s, \phi_0, b_0 ; e^i) = w_t(s, e^i), \tag{1.14}$$

where $e^i = (0,..., 0, 1, 0,..., 0)$ is the i-th unit vector in R^k, and $(s, \phi_0, b_0) \in N_1$, then $\partial x(t, s, \phi_0, b_0)/\partial b^i$ satisfies Eq. (1.13) with $\Psi_0 = b_0$, $\beta = e^i$, which may be written, in more familiar notation as

$$\dot{w}(t) = \frac{\partial f}{\partial \phi}(t, x_t(s, \phi_0, b_0)(w_t) + \frac{\partial f}{\partial b^i}(t, x_t(s, \phi_0, b_0); b_0), \tag{1.15}$$

with $w_s = 0$.

The meaning of $\partial f/\partial \phi$ in (1.15) is clear, upon comparison with (1.13), and we have used $(\partial f/\partial b^i)(t, x_t(s, \phi_0, b_0), b_0)$ to denote the element in R^n given by $d_2 f(t, x_t(s, \phi_0, b_0), b_0; e^i)$; using the same reasoning as in (1.14). Further, for $b \in R^k$ and $w_t(s, b)$, the solution of (1.13) with $\Psi = b_0$, $\beta = b$, satisfies

$$w_t(s, b) = \sum_{i=1}^{k} b^i \frac{\partial x}{\partial b^i}(t, s, \phi_0, b_0), \tag{1.16}$$

the multiplication on the right in (1.16) being scalar multiplication in C. Thus, in this special case, every solution of (1.13) may be written as a linear combination of the k solutions of (1.15), $\{\partial x(t, s, \phi_0, b_0)/\partial b^i\}_{i=1}^k$.

COROLLARY 1.14. *If, in addition to the hypotheses of Corollary* 1.12, *it is assumed that $f \in C^2$ on N_1 (i.e., the mappings $d_1 : N_1 \to L(C, R^n)$ and $d_2 : N_1 \to L(E, R^n)$ are each continuously differentiable on N_1), then $x_t(s, \phi_0, \Psi_0) \in C^2$ on the domain $s \leqslant t < s + A$, $(s, \phi_0, \Psi_0) \in N_1$.*

Proof. Since $d_1 x(t, s, \phi_0, \Psi_0; \alpha)$ is again a solution of the FDE (1.6) (with the added variable Ψ_0) and $d_2 x(t, s, \phi_0, \Psi_0; \beta)$ is a solution of the FDE (1.13), the corollary is an immediate consequence of applying Corollary 1.12 to Eqs. (1.6) and (1.13).

2. In this section, the definitions and terminology of Ref. [6] will be used. Consider the FDE

$$\dot{y}(t) = f(y_t), \tag{2.1}$$

where f maps M, an open convex subset of C into R^n, and $f \in C^2$ on M. Assume that (2.1) has a $(k + 1)$-parameter family of periodic solutions contained in M, $\{y^0(t, b)\}$ for $b \in U$, U an open subset of R^k, with

$y^0(t + \tau(b); b) = y^0(t, b)$ for all $b \in U$, with the period $\tau(b) > 0$. We will suppose that as a function of b, $\tau(b)$, and $y^0(t, b) \in C^2$ on U.

These assumptions imply that the linear variational equation associated with the family of periodic solutions of (2.1),

$$\dot{x}(t) = df(y_t^0(b); x_t) \tag{2.2}$$

has, for each $b \in U$, 1 as a characteristic multiplier of multiplicity $k + 1$, if the $(k + 1)$-partial derivatives $\{\partial y^0(t, b)/\partial t,\ \partial y^0(t, b)/\partial b^j,\ j = 1,..., k\}$ are independent functions. But, at a point $b \in U$ for which the vector $\partial\tau(b)/\partial b \neq 0$, then the period map associated with (2.2) has only k independent eigenfunctions corresponding to the eigenvalue 1. This is equivalent to asserting that (2.2) has only k independent periodic solutions. Thus, from the assumption above, there is one solution which is $0(1 + |t|)$ for all t, that is, the initial condition of this solution lies in the nullity of $(I - T)^2$, but not in the nullity of $(I - T)$, where T is the period map of (2.2), and I the identity map on C. This necessitates discussing the orbital stability of solutions of (2.1), with respect to the family $\{y^0(t, b)\}$, $b \in U$.

The following definition is based on those given in Hale and Stokes [3].

Let $V(b) = \{y_t^0(b) : 0 \leqslant t \leqslant \tau(b)$ denote the trajectory of $y^0(t, b)$ in C for each $b \in U$. For $U^* \subset U$, let $V^* = \cup\{V(b), b \in U^*\}$. If $U^* = U$, we write V for V^*. V is the integral manifold of (2.1) defined by $y^0(t, b)$ in C, $-\infty < t < +\infty$ as b varies over U.

Definition 2.1. We will say that the manifold V of solutions of (2.1) is strongly orbitally stable (orbitally asymptotically stable with asymptotic amplitude and asymptotic phase in [3]), if given $U^* \subset U$, U^* open, relatively compact, and $\bar{U}^* \subset U$, we have: (i) For any $\nu > 0$, there exists a $\delta > 0$ such that if ϕ is in a δ neighborhood of V^*, then $y_t(\phi)$, the corresponding solution of (2.1), with $y_0(\phi) = \phi$, is in an ν neighborhood of V for all $t \geqslant 0$. (ii) There exists a ρ_0 such that for any $\phi \in C$, $\text{dist}(\phi, V^*) < \rho_0$, then $\| y_t(\phi) - y^0_{t+c}(b_0)\| \to 0$ as $t \to \infty$, for some $b_0 \in U$, $c \in R$. Here dist (ϕ, V^*) denotes the distance from a point to a set in the C-norm.

With these preliminaries, we may state:

Theorem 2.2. *Assume Eq.* (2.1) *has a* $(k + 1)$*-parameter family of period solutions* $y^0(t, b)$ *of period* $\tau(b) > 0$ *for* $b \in U$, *an open subset of* R^k, *where* $\tau(b), y^0(t, b) \in C^2(b)$ *on* U. *If, for every* $b \in U$, *the linear variational equation* (2.2) *has* 1 *as a characteristic multiplier of multiplicity* $(k + 1)$, *and the remaining characteristic exponents all have negative real parts, then the integral manifold* V, *is strongly orbitally stable.*

Note. The definitions of characteristic multiplier, multiplicity, simple elementary divisors, and characteristic exponents are defined in Lemma 2.2 of Ref. [6] (Stokes).

Remark. The assumption that 1 has multiplicity $(k+1)$ for every $b \in U$ is a nondegeneracy condition which is the exact counterpart of the condition in Hale and Stokes [3], which, for ordinary differential equations, states that the matrix of functions $(\partial y^0(t, b)/\partial t, \partial y^0(t, b)/\partial b)$ be of rank $(k+1)$ for all $b \in U$.

The proof of Theorem 2.2 is in two sections. Section 3 shows that for each $b \in U$, there exists a stable manifold $R^*(b)$, with the property that $\phi \in R^*(b)$ implies $\| y_t(\phi) - y_t^0(b)\| \to 0$ as $t \to +\infty$, (and dist $(y_t^0(b), y_t(\phi)) \to 0$ as $\| \phi - y_0^0(b)\| \to 0$, uniformly for $t \geqslant 0$). Section 4 then proves that given $U^* \subset U$, U^* open, relatively compact, $\bar{U}^* \subset U$, then for some $\rho_0 > 0$, $\operatorname{dist}(\phi, V^*) < \rho_0$ implies $y_s(\phi) \in R^*(b_0)$ for some $b_0 \in U$, $s \geqslant 0$. Thus, $\| y_{t+s}(\phi) - y_t^0(b_0)\| = \| y_{t'}(\phi) - y^0_{t'+c}(b_0)\| \to 0$ as $t' \to +\infty$, where $t' = t + s$, $c = -s$. This will prove Theorem 2.2, and incidentally show that the set $R^* = \cup \{R^*(b), b \in U\}$ is transverse to the flow in C defined by (2.1), in some neighborhood of V.

3. We wish to consider Eq. (2.1) in the form

$$\dot{z}(t) = df(y_t^0(b); z_t) + g(t, z_t, b), \tag{3.1}$$

where $z_t = y_t - y_t^0(b)$, and g is related to f as in (1.1).

The hypotheses of Theorem 2.2 imply that the linear variational equation

$$\dot{x}(t) = df(y_t^0(b) : x_t) \tag{3.2}$$

has the following properties, for every $b \in U$, stated as a lemma for reference.

LEMMA 3.1. (i) *There exists two closed subspaces of C, $E(b, s)$, and $K(b, s)$, for every s, with C as their direct sum, and positive $B = B(b)$ and $\sigma = \sigma(b)$, continuous in b, such that if $\phi \in K(b, s)$, then $\| x_t(\phi, s)\| \leqslant Be^{-\sigma(t-s)}\| \phi \|$ for $t \geqslant s$, where $x_t(\phi, s)$ is the solution of* (3.2), *$x_s(\phi, s) = \phi$. Further, dimension $E(b, s) = k + 1$, and if $\phi \in E(b, s)$, the corresponding solution $x_t(\phi, s)$ of* (3.2) *is defined and satisfies $\| x_t(\phi, s)\| = 0(1 + | t - s |)$ for all t and s.*

(ii) *If $X(t, s, b)$ denotes the matrix solution of* (3.2) *with $X(s, s, b) = I$, the identity matrix, and $X(t, s, b) = 0$, $s - h \leqslant t \leqslant s$ (see Ref.* [6] *for remarks concerning the existence of X), then there exists two matrices $X_1(t, s, b)$ and $X_2(t, s, b)$, each a matrix solution of* (3.2) *with $X(t, s, b) = X_1(t, s, b) + X_2(t, s, b)$, for $t \geqslant s - h$, such that X_1 and X_2 have the properties: X_1 is defined for all $t, s \geqslant 0$ and $X_2(t, s, b)$ is defined for $0 \leqslant s \leqslant t$, and both are locally integrable in t and s. Further, there exist constants B_1, $\sigma > 0$, depending upon b, such that*

$\| X_1(t, s, b)\| \leqslant B_1(1 + | s - t |)$, $0 \leqslant t \leqslant s$, *and* $\| X_2(t, s, b)\| \leqslant B_1 e^{-\sigma(t-s)}$, $t \geqslant s \geqslant 0$.

(iii) *Every bounded solution of the integral equation*

$$z(t) = x(t, \phi) + \int_0^t X_2(t, s, b)\, g(s, z_s, b)\, ds$$
$$- \int_t^\infty X_1(t, s, b)\, g(s, z_s, b)\, ds, \qquad t \geqslant 0,$$
$$x(t) = \phi(t) - \int_0^\infty X_1(t, s, b)\, g(s, z_s, b)\, ds, \qquad -h \leqslant t \leqslant 0, \tag{3.3}$$

where $\phi \in C$, $x(t, \phi)$ *satisfies* (3.2) *with* $x_0(\phi) = \phi$, *is also a solution of* (3.1), *for* $t \geqslant 0$.

Proof. Referring to Ref. [6], most of the statements in (i) follow from Lemma 2.2, (ii) is a consequence of Lemma 3.2, and (iii) is a restatement of Lemma 3.3. The continuity of $B_1(b)$ and $\sigma(b)$ in b follow from the remark that $\max\{| \lambda | : \lambda \neq 1$, λ in the spectrum of the period map $T(b)$ of (3.2)$\}$ is continuous in b.

By a slight extension of Lemmas 3.4 and 3.5 of the same paper, we obtain:

LEMMA 3.2. *Assume* (3.1) *and* (3.2) *satisfy the hypotheses of Theorem* 2.2. *Choose* a, *with* $0 < a < 1$. *Then there exist a positive function* $\eta_0(b)$, *nonnegative functions* $\rho_1(\eta, b)$ *and* $\rho_2(\eta, b)$, *defined and continuous for* $b \in U$, $0 \leqslant \eta \leqslant \eta_0(b)$, *with* ρ_1 *and* ρ_2 *increasing in* η *for* b *fixed*; $\rho_1(0, b) = 0 = \rho_2(0, b)$, $\rho_2(\eta_0(b), b) < 1$, *such that given* $\phi \in K(b, 0)$, $\| \phi \| < \eta_0(b)$; *there exists a solution* $z^*(\phi, b)$ *of* (3.3) *satisfying*

$$\| z_t^*(\phi, b)\| \leqslant \rho_1(\| \phi \|, b)\, e^{-a\sigma t}, \qquad t \geqslant 0, \tag{3.4}$$

where $\sigma = \sigma(b)$ *is given in Lemma* 3.1(i). *Further, if we define the mapping* $H(\phi, b)$ *into* C, *for* $b \in U$, $\phi \in K(b, 0)$, *and* $\| \phi \| < \eta_0(b)$ *by*

$$H(\phi, b)(\theta) = \int_0^\infty X_1(\theta, x)\, g(s, z_s^*(\phi, b), b)\, ds, \qquad -h \leqslant \theta \leqslant 0. \tag{3.5}$$

then

$$\| H(\phi_1, b) - H(\phi_2, b)\| \leqslant \rho_2(\max(\| \phi_1 \|, \| \phi_2 \|), b)\| \phi_1 - \phi_2 \|. \tag{3.6}$$

Proof. The only change needed in the proof as given in Ref. [6] involves the modified estimate of the integral involving $X_1(t, s, b)$, which occurs in Ref. [6, p. 133]. But if $\| z_t \| \leqslant e^{-\alpha t} \| p \|$, then

$$\left\| \int_{t+\theta}^{\infty} X_1(t+\theta, s, b)\, g(s, z_s)\, ds \right\| \leqslant \| p \| \cdot \epsilon_0 B_1 \int_{t+\theta}^{\infty} e^{-\alpha s}[1 + s - (t+\theta))\, ds$$

$$= | p | \epsilon_0 B_1 \left[\frac{2}{\alpha}\right] e^{-\alpha(t+\theta)},$$

and the addition of the term $1/\alpha$ does not affect the subsequent proof.

Remark. From the continuity properties with respect to b in the above statement, it is immediate that if b is restricted to a relatively compact set U^*, the estimates in (3.4) and (3.6) can be made independent of b.

From the two lemmas preceding, we obtain the desired statement concerning the existence of the stable manifold $R^*(b)$.

THEOREM 3.3. *Assume* (3.2) *has* 1 *as a characteristic multiplier, with the dimension of* $E(b, 0) = k + 1$, *and that the remaining characteristic exponents of* (3.2) *have negative real parts, for every* $b \in U$. *Then there exists a mapping* $R(\phi, b)$ *into* C, *for* $b \in U, \phi \in K(b, 0), \| \phi \| < \eta_0(b)$, *given by* $R(\phi, b) = \phi - H(\phi, b)$, $\eta_0(b)$ *defined in Lemma* 3.2, H *by* (3.5), *with the following properties*: (i) *If* $z_t(\phi, b)$ *is the solution of* (3.1) *with* $z_0(\phi, b) = R(\phi, b)$; *then* $z_t(\phi, b) \to 0$ *exponentially as* $t \to \infty$. (ii) $R(\cdot, b)$ *is a homeomorphism of the ball* $S(\eta_0, b) = \{\phi \in K(b, 0) : \| \phi \| < \eta_0(b)\}$ *onto* $R(S(\eta_0, b)b)$. (iii) $R(\phi, b)$ *is Fréchet differentiable in* ϕ *and* b *for* $b \in U$ *at* $\phi = 0$.

Proof. Properity (i) follows from Lemmas 3.1 and 3.2. The proof that $R(\cdot, b)$ is a homeomorphism is staightforward, using the inequality $\rho_2(\eta_0(b); b) < 1$ in (3.6). For (iii), we shall show that $\partial R/\partial\phi\, (0, b; \Psi) = \Psi$, and $\partial R/\partial b\, (0, b; a) = 0$, $\Psi \in C$, $a \in R^k$, for all $b \in U$. Now $\partial R/\partial \phi\, (0; \Psi) = \Psi$ is equivalent to asserting that $H(\Psi, b) = o(\| \Psi \|)$ as $\| \Psi \| \to 0$, for all $b \in U$. But this follows from $\| H(\Psi, b)\| \leqslant \rho_2(\| \Psi \|, b) \cdot \| \Psi \|$, for $\| \Psi \| < \eta_0(b)$, and $\rho_2(0, b) = 0$. $\partial R/\partial b\, (0, b; a) = 0$ is immediate as $H(0, b) = 0$ for all $b \in U$. This completes the proof of Theorem 3.3.

Note that in proving (iii) above, the following result concerning the differentiability of H was obtained, stated as a corollary for reference.

COROLLARY 3.4. *The mapping* $H(\phi, b)$ *defined by* (3.5) *is Fréchet differentiable in* ϕ *and* b *for* $b \in U$ *at* $\phi = 0$, *and* $\partial H/\partial \phi\, (0, b; \cdot) = 0$, $\partial H/\partial b\, (0, b; \cdot) = 0$ *for all* $b \in U$.

Let $P(b)$ denote the projection onto the space $K(b, 0)$. The following lemma follows from Dieudonné ([2], p. 261, 10.1.3), in precisely the same manner as did Corollary 3.8 of Ref. [6].

LEMMA 3.5. *Let* $H_1(\phi, b) = P(b)H(\phi, b)$, *and* $R_1(\phi, b) = \phi - H_1(\phi, b)$. *Then there exists an open neighborhood of* 0 *in* $K(b, 0)$, $W \subset S(\eta_0, b)$ *such that* R_1 *is a homeomorphism of* W *onto* $S(\eta_0/2, b)$.

4. To complete the proof of Theorem 2.2, we wish to show that given any $U^* \subset U$, U^* open and relatively compact, with $\bar{U}^* \subset U$, there exists a ζ such that if $\operatorname{dist}(\phi, V^*) < \zeta$, then for some time $r > 0$ and some $b \in U$, $y_r(\phi) \in y_0^0(b) + R(S(\eta_0, b), b)$, as then the results of Section 3 imply that $\| y_{t+r}(\phi) - y_t^0(b)\| \to 0$ exponentially as $t \to +\infty$. As noted in the remark following Lemma 3.2, from the compactness of $\bar{U}^*$, it follows that it suffices to show that for *each* $b_0 \in U^*$ there exists a $\zeta(b_0) > 0$, such that if $\| \phi - y_0^0(b_0)\| < \zeta$, then for some $r > 0$, and some $b \in U$,

$$y_r(\phi) \in y_0^0(b) + R(S(\eta_0, b), b).$$

From the continuity with respect to initial conditions of solutions of (2.2), it follows that given any integer m, and any $b_0 \in U^*$, there exists a $\zeta_0 > 0$, such that if $\| \phi - y_0^0(b_0)\| < \zeta_0$, then $y_t(\phi)$ is defined on $[0, (m+1)\tau(b_0)]$, and on this interval $\| y_t(\phi) - y_t^0(b)\|$ may be made as small as desired, for all b satisfying $\| b - b_0 \| \leqslant \zeta_0$. In particular, we may take m so large that $\dot{y}_t(\phi)$ exists on $[(m-1)\tau(b_0), (m+1)\tau(b_0)]$, and $\| y_t(\phi) - y_t^0(b)\|$ so small that on some interval $(m\tau(b_0) - \zeta_1, m\tau(b_0) + \zeta_1)$, $0 < \zeta_1 < \tau(b_0)$, $[P(b)(y_t(\phi) - y_t^0(b))]$ is in the domain of $R_1^{-1}(\cdot, b)$, that is,

$$\| P(b)(y_t(\phi) - y_t^0(b))\| < \eta_0(b)/2 \tag{4.1}$$

for $t \in (m\tau(b_0) - \zeta_1, m\tau(b_0) + \zeta_1)$, $\| \phi - y_0^0(b_0)\| < \zeta_0$ for all b with $\| b - b_0 \| \leqslant \zeta_0$. (See Lemma 3.5.)

For convenience of notation, replace t by $t - m\tau(b_0)$, so now $y_t(\phi)$ and $\dot{y}_t(\phi)$ are defined on $(-\zeta_1, \zeta_1)$, and satisfy (4.1) on this interval. Now, we wish to find b, $\| b - b_0 \| \leqslant \zeta_0$, $\Psi \in S(\eta_0, b) \subset P(b)C$ and $r \in (-\zeta_1, \zeta_1)$ such that

$$y_r(\phi) - y_0^0(b) = R(\Psi, b). \tag{4.2}$$

Applying the projections $P(b)$ and $Q(b) = I - P(b)$, (4.2) is equivalent to the assertion that there exists b, $\| b - b_0 \| \leqslant \zeta_0$, $\Psi \in S(\eta_0, b)$, and $r \in (-\zeta_1, \zeta_1)$ such that

$$P(b)(y_r(\phi) - \gamma_0^0(b)) = R_1(\Psi, b), \tag{4.3a}$$

$$Q(b)(y_r(\phi) - \gamma_0^0(b)) = Q(b)\, R(\Psi, b) = Q(b)\, H(\Psi, b) \stackrel{\text{def}}{\equiv} H_2(\Psi, b). \tag{4.3b}$$

But from (4.1), (4.3a) may be solved for Ψ, that is

$$\Psi(r, b, \phi) = R_1^{-1}(P(b)(y_r(\phi) - y_0^0(b), b),$$

for all $r \in (-\zeta_1, \zeta_1)$ $\| b - b_0 \| \leqslant \zeta_0$, $\| \phi - y_0{}^0(b_0) \| < \zeta_0$. Thus it suffices to determine r and b for each ϕ (in the above sets), such that

$$Q(b)(y_r(\phi) - y_0{}^0(b)) - H_2(\Psi(r, b, \phi), b) = 0. \tag{4.4}$$

Denote the left side of (4.4) by $F(r, b, \phi)$.

Now Eq. (4.4) possesses a solution for $\phi = y_0{}^0(b_0)$, namely $r = 0$, $b = b_0$, as $H_2(\Psi(0, b_0, y_0{}^0(b_0)), b_0) = H_2(0, b_0) = 0$. Thus, if the Fréchet derivative of $F(r, b, \phi)$ with respect to the $(k + 1)$-vector (r, b) at $r = 0$, $b = b_0$, and $\phi = y_0{}^0(b_0)$ is a linear homeomorphism of R^{k+1} onto $Q(b_0)C$, then the standard implicit function theorem in Banach spaces (see Dieudonné [2]), implies that for each ϕ in some neighborhood of $y_0{}^0(b_0)$, there exists $r(\phi)$ and $b(\phi)$ such that $F(r(\phi), b(\phi), \phi) = 0$.

It should be remarked that $F(r, b, \phi)$ is differentiable with respect to (r, b, ϕ), for $-\zeta_1 < r < \zeta_1$, $\| b - b_0 \| < \zeta_0$, and $\| \phi - y_0{}^0(b_0) \| < \zeta_0$. This follows directly from the assumptions in Theorem 2.2 that $f \in C^2$ on $M \subset C$, and $\tau(b), y_t{}^0(b) \in C^2(b)$ on V. These imply (by Theorem 1.11 and Corollary 1.14) that $y_r(\phi) \in C^2(\phi)$ (and $y_r(\phi) \in C^1(r)$ by our choice of ζ_1), and that $T(b) \in C^1(b)$, where $T(b)$ denotes the period map defined by the linear variational equation (3.2). As Theorem 2.2 assumes that 1 is an isolated point of the spectrum of $T(b)$, the mapping $Q(b)$, the projection onto the corresponding eigenspace $E(b, 0)$ may be defined in terms of the integral of $[T(b) - I]^{-1}$ on some contour in the complex plane enclosing 1 and no other points of $\sigma(T(b))$. From standard theorems on Fréchet derivatives, it follows that $Q(b) \in C^1(b)$ and $P(b) = (I - Q(b)) \in C^1(b)$, also. Theorem 3.3 established that $R(\Psi, b) \in C^1(\Psi, b)$. As the composition of differentiable maps is differentiable, it follows $F(r, b, \phi) \in C^1(r, b, \phi)$. So all that remains is to evaluate the derivative of $F(r, b, \phi)$ with respect to r and b at $I_0 = (0, b_0, y_0{}^0(b_0))$.

To compute the r derivative at I_0, note that

$$\frac{\partial}{\partial r}(y_r(\phi) - y_0{}^0(b)) = \dot{y}_0{}^0(b_0) \in Q(b_0)\, C.$$

Accordingly,

$$\frac{\partial}{\partial r}[P(b)(y_r(\phi) - y_0{}^0(b))] = 0$$

at I_0. From this, it follows readily, using Corollary 3.4, that

$$\partial/\partial r\, H_2(\Psi(r, b, \phi), b) = 0$$

at I_0. Thus,

$$\frac{\partial}{\partial r} F(r, b, \phi) = Q(b_0)\dot{y}_0{}^0(b_0) \qquad \text{at} \quad I_0. \tag{4.5}$$

To compute the b derivative of F, first note that

$$\frac{\partial}{\partial b}(y_r(\phi) - y_0{}^0(b)) = -\frac{\partial}{\partial b} y_0{}^0(b) \in Q(b)\, C,$$

so

$$\frac{\partial}{\partial b}[P(b)(y_r(\phi) - y_0{}^0(b))] = 0 \qquad \text{at} \quad I_0 .$$

From this we obtain $(\partial/\partial b)\Psi(r, b, \phi) = 0$ at I_0, as $\Psi(0, b_0, y_0{}^0(b_0)) = 0$. Also, it follows from Corollary 3.4 that $(\partial/\partial b)H_2(\Psi(r, b, \phi), b) = 0$ at I_0.

Thus, all that remains is to evaluate the b derivative of $Q(b_0)(y_r(\phi) - y_0{}^0(b))$ at I_0. Again, $y_r(\phi) - y_0{}^0(b) = 0$ at I_0 implies that

$$\frac{\partial}{\partial b}[Q(b)(y_r(\phi) - y_0{}^0(b))] = Q(b)\frac{\partial}{\partial b}(y_r(\phi) - y_0{}^0(b)) = -Q(b_0)\frac{\partial}{\partial b} y_0{}^0(b_0) \quad \text{at } I_0. \tag{4.6}$$

The assumption in Theorem 2.2 that 1 is a characteristic multiplier of multiplicity $k + 1$ for every $b \in U$ imply that $(\dot{y}_0{}^0(b), \partial y_0{}^0(b_0)/\partial b)$ form a basis for $Q(b)C$, for every $b \in U$. But (4.5) and (4.6) show that the linear mapping from R^{k+1} into $Q(b_0)C$ given by the (r, b)-derivative of F at I_0 is defined by the mapping which takes $t \in R$ and $a \in R^k$ into

$$Q(b_0)\left[t\dot{y}_0{}^0(b_0) - \sum_{i=1}^{k} a^i \frac{\partial y_0{}^0(b_0)}{\partial b^i}\right] = t\dot{y}_0{}^0(b_0) - \sum_{i=1}^{k} a^i \frac{\partial y_0{}^0(b_0)}{\partial b^i},$$

as $Q(b_0)$ is the identity on $Q(b_0)C$. And since $(\dot{y}_0{}^0(b_0), \partial y_0{}^0(b_0)/\partial b)$ is a basis in $Q(b_0)C$, this mapping has a unique linear inverse, which is continuous, as the spaces are finite dimensional.

So the hypotheses of the implicit function theorem are satisifed, and the desired $r(\phi)$, $b(\phi)$ then exist, that is (4.2) has a solution. This completes the proof of Theorem 2.2.

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